On Countable Completions of Quotient Ordered Semigroups

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Abstract

A poset is said to be $\boldsymbol{\omega}$-chain complete if every countable chain in it has a least upper bound. It is known that every partially ordered set has a natural $\boldsymbol{\omega}$-completion. In this paper we study the $\boldsymbol{\omega}$-completion of partially ordered semigroups, and the topological action of such a semigroup on its $\boldsymbol{\omega}$-completion. We show that, for partially ordered semigroups, $\boldsymbol{\omega}$-completion and quotient with respect to congruences are two operations that commute with each other. This contrasts with the case of general partially ordered sets.

1 Introduction

Completions of partial orders and lattices have been extensively analyzed by several authors. Different kinds of completions have been introduced (see [3] for a survey): ideal completion, completion based on generalizations of Dedekind cuts [5]—the most famous being of course the original completion of rational numbers by real numbers—, MacNeille completion that embeds a poset into a lattice [1], are examples of them. A completion technique based on the use of closure operators was introduced in [7] to complete a poset into either a chain-complete poset or into a DCPO (Directed Complete POset). It has the advantage of adding as few points as necessary to get, say, a DCPO. Hence, the completion of a DCPO by this method is naturally isomorphic to the original DCPO. Considered within the appropriate categories, this completion provides the left member of a reflection between posets and DCPOs (or between posets and chain-complete posets), the right member of which is the forgetful functor.

In this paper we study the connection between semigroups and partial orders from the order completion point of view. More precisely, we are interested in the $\boldsymbol{\omega}$-chain completion of partially ordered semigroups. A $\boldsymbol{\omega}$-chain completion, or $\boldsymbol{\omega}$-completion, of a poset P is a universal poset Q such that P is embedded in Q, and Q is such that any countable chain in Q has a least upper bound. Among partially ordered semigroups, we pay a particular interest to *naturally ordered semigroups*, *i.e.*, semigroups that are ordered by the collapse partial order obtained from the left divisibility relation; so that for two elements x and

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y of a semigroup S, we have $x \preccurlyeq y$ if there is an element $r \in S$ such that $y = x \cdot r$. Since semigroups are very often presented through generators and relations, we consider the following situation: let S be a naturally ordered semigroup, and let Q be its completion. Consider furthermore a congruence $\sim$ on S, that is to say, an equivalence relation compatible with the semigroup structure. The quotient $S/\sim$ yields another semigroup, again preordered by the divisibility relation, and thus a partially ordered semigroup G through its collapse. If H is the $\boldsymbol{\omega}$-completion of G, we obtain thus the commutative square:

$$\begin{tikzcd} S \arrow[r] \arrow[d] & Q \arrow[d] \\ G \arrow[r] & H \end{tikzcd}$$

The first question we are interested in is the following: Can the partial order H be given a description as a quotient of the partial order Q? In other words, “does the quotient of the completion coincide with the completion of the quotient?” The question is not trivial since, for general partial orders, the answer is *no.* But it happens to be true for some naturally ordered semigroups.

We give two applications of the universal property of the completion. The first one is the construction of a natural topological semigroup action of a semigroup on its completion. The topology we consider on the completion is the $\boldsymbol{\omega}$-Scott topology, a topology that mimics the Scott topology when only countable directed sets have least upper bounds. We say our base semigroup is *strongly ordered* when the partial order is a both a left and a right congruence. Our second application of the universal property of the completion is the construction of a topological semigroup structure on the completion of a strongly ordered semigroup, that extends the semigroup action of the original semigroup.

The paper is organized as follows. Sections 2, 3 and 4 recall some background on completion of posets, quotients of posets and on partially ordered semigroups. Sections 5 to 7 contain our main results on $\boldsymbol{\omega}$-completion of quotient semigroups, the continuous action of a semigroup on its completion and the completion of this continuous action into a topological semigroup structure.

2 Completion of Partial Orders

2.1 The categories we consider. Let $(P, \leq)$ be a *poset* (partially ordered set). A *directed* subset of P is any subset D of P such that any two elements of D have a common upper bound in D. If D is a directed subset of P, a subset D' of D is said to be *cofinal* in D if, for any element x of D, there exists an element y of D' such that $x \leq y$. Subset D' is then directed itself. Poset P is said to be a Dcpo (Directed Complete POset) if any directed subset of P has a least upper bound (*lub*) in P.

A subset C of P is called a *chain* if it is totally ordered by the restriction of the order of P. Chains are directed subsets of P. A countable chain is called a $\boldsymbol{\omega}$-chain. We say that poset P is *chain complete* if any chain of P has a *lub* in P. It is a known result (Iwamura's lemma, see *e.g.* [2]) that a poset is chain complete if and only if it is a DCPO. We say that a poset P is *$\boldsymbol{\omega}$-chain complete* if any $\boldsymbol{\omega}$-chain of P has a *lub* in P. Since it is clear that any countable directed subset of P has a cofinal $\boldsymbol{\omega}$-chain, poset P is $\boldsymbol{\omega}$-chain complete if and only if any countable directed subset of P has a *lub* in P—actually, this is also true although less obvious if $\boldsymbol{\omega}$ is replaced by any infinite cardinal [7].

Let $h : P \to Q$ be a mapping between two posets P and Q. We say h is *increasing* if, for any two elements x and y of P, we have $x \leq y \Rightarrow f(x) \leq f(y)$. In this case, the subset $f(C)$ is a chain (respectively, a $\boldsymbol{\omega}$-chain) of Q whenever C is a chain (respectively, a $\boldsymbol{\omega}$-chain) of P. We say that h is $\boldsymbol{\omega}$-continuous if h is increasing, and if for any $\boldsymbol{\omega}$-chain C of P, if C has a *lub*, then $f(C)$ has a *lub* and $\sup f(C) = f(\sup C)$ holds. We denote by PO the category with posets as objects, and with increasing mappings as morphisms, and we denote by ω-PO the category with $\boldsymbol{\omega}$-chain complete posets as objects and $\boldsymbol{\omega}$-continuous mappings as morphisms.

2.2 Completion of posets. We refer to [6, Ch. 4] for definitions and properties related to adjunction pairs, universal arrows and reflections.

A convenient way of defining the completion of a poset is to express it as a universal arrow from an adequate category of posets, to the forgetful functor. For this, we need to slightly alter the category PO. We define PO' as the category with posets as objects, and with $\boldsymbol{\omega}$-continuous mappings between them as morphisms.

A *$\boldsymbol{\omega}$-completion* of a poset P will be given as a pair (Q, f), where Q is an object in ω-PO and $f : P \to Q$ is a morphism of PO' which is a universal arrow from P to the forgetful functor ω-$\mathsf{PO} \to \mathsf{PO}'$. In other words, the pair (Q, f), if it exists, has the property that for any object R in ω-PO and for any morphism $g : P \to R$ of PO', there is a unique morphism $h : Q \to R$ of ω-PO making the following diagram commutative:

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[dr, "g"'] & Q \arrow[d, "h"] \\ & R \end{tikzcd}$$

As usual, this makes the pair (Q, f) unique, up to a unique isomorphism, if it exists. The existence of $\boldsymbol{\omega}$-completions is guaranteed by the following result, adapted from [7].

2.3 Theorem (Markowsky). — *The forgetful functor* $\mathbf{U} : \omega\text{-}\mathsf{PO} \to \mathsf{PO}'$ *is the right adjoint of an adjunction pair* $(\mathbf{R}, \mathbf{U})$. *Hence for each object P of* PO', *if Q denotes the object $Q = \mathbf{R}(P)$ of* ω-PO, *there is a natural morphism $f : P \to Q$, which is an embedding of posets, and the pair (Q, f) is universal from P to $\mathbf{U}$. Furthermore poset P is join-dense in Q, which means that the* *following holds:*

$$\forall x \in Q, \quad x = \sup_Q \{f(z) \,:\, z \in P,\, f(z) \leq x\}.$$

Finally, the adjunction pair $(\mathbf{R}, \mathbf{U})$ *is a reflection. Hence if* P *is already* $\boldsymbol{\omega}$*-chain complete, then* $f : P \to Q$ *is an isomorphism.*

2.4 Notations and conventions. According to Theorem 2.3, the natural mapping $f : P \to Q$ from P to any of its completion is an embedding. We will thus simplify the notations by suppressing f, and simply assume that $P \subseteq Q$. Furthermore, we will freely talk of *the* $\boldsymbol{\omega}$-completion of a poset, since $\boldsymbol{\omega}$-completions are unique up to a unique isomorphism.

2.5 Strict completions. Topological considerations originating from domain theory are quite unavoidable when studying "continuous" posets, in a wide sense. Since we are specially interested with countable cardinality, we adapt the fundamental notion of compact element to our needs through the following definitions.

2.6 Definition. — *Let* P *be a* $\boldsymbol{\omega}$*-chain complete poset. An element* x *of* P *is said to be* $\boldsymbol{\omega}$*-compact whenever, for any* $\boldsymbol{\omega}$*-chain* C *of* P*, if* $x \leq \sup C$*, there exists an element* $c \in C$ *such that* $x \leq c$*.*

2.7 Definition. — *Let* P *be a poset. We say that the* $\boldsymbol{\omega}$*-completion* $P \to Q$ *of* P *is* strict*, or simply that poset* P *is a* strict poset*, if any element* $x \in P$ *is* $\boldsymbol{\omega}$*-compact in* Q*, and if for every element* $y \in Q$*, there exists a* $\boldsymbol{\omega}$*-chain* $C \subseteq P$ *such that* $y = \sup_Q C$*.*

Not every poset has a strict completion. Consider for example the poset P given by $P = \{0, 1, 2, \ldots\} \cup \{\infty\}$ equipped with the natural order. Then the completion $Q = \mathbf{R}(P)$ is isomorphic to P since P is $\boldsymbol{\omega}$-chain complete. But the image $f(\infty)$ is not $\boldsymbol{\omega}$-compact in Q, where $f : P \to Q$ denotes the natural isomorphism.

2.8 $\boldsymbol{\omega}$-Scott topology. Scott topology cannot be defined as usual for $\boldsymbol{\omega}$-chain complete posets. Instead we will use the following variant of Scott topology. Let $(P, \leq)$ be a $\boldsymbol{\omega}$-chain complete poset. We say that a subset $U \subseteq P$ is *$\boldsymbol{\omega}$-Scott open* if U is an upper set (*i.e.*, for any $x, y \in P$, if $x \in U$ and if $x \leq y$, then $y \in U$), and if U satisfies the following property: for any $\boldsymbol{\omega}$-chain $C \subseteq P$, if $\sup C \in U$, then $C \cap U \neq \emptyset$. The family of $\boldsymbol{\omega}$-Scott open subsets of P forms a topology on P.

2.9 Relation with $\boldsymbol{\omega}$-continuous mappings and strict completions. If P and Q are two $\boldsymbol{\omega}$-chain complete posets, usual techniques (see *e.g.* [4]) show that a mapping $f : P \to Q$ is $\boldsymbol{\omega}$-continuous in the sense of § 2.1 if and only if f is continuous with respect to $\boldsymbol{\omega}$-Scott topologies on P and Q respectively.

Say that $\boldsymbol{\omega}$-chain complete poset P is $\boldsymbol{\omega}$-algebraic if any element $x \in P$ is the *lub* of a ω-chain of ω-compact elements of P. Usual techniques from domain theory show that for such a poset, ω-Scott topology is generated by subsets of the form $\uparrow x = \{y \in P \,:\, x \leq y\}$, for x ranging over ω-compact elements of P. This in particular the case for strict ω-completions.

3 Quotients of Partial Orders

3.1 Preorders and orders. Recall that a preorder is a pair $(P, \leq)$, where P is a set and $\leq$ is a reflexive and transitive binary relation on P. If PreO denotes the category with preorders as objects, and increasing mappings as morphisms, with the same definition as for posets, the forgetful functor $\mathbf{V} : \mathsf{PO} \to \mathsf{PreO}$ is the right member of an adjunction pair $(\mathbf{P}, \mathbf{V})$. If $(P, \leq)$ is a preorder, the poset $\mathbf{P}(P)$ is called the *collapse* of P, and it comes with a natural morphism of preorders $P \to \mathbf{P}(P)$. Furthermore, the adjunction pair $(\mathbf{P}, \mathbf{V})$ is a reflection.

3.2 Quotient of a poset. Let $(P, \leq)$ be a poset, and let $\sim$ be an equivalence relation on P. We call *quotient* of P with respect to $\sim$ any pair (T, f), where $f : P \to T$ is a morphism of posets which is constant on $\sim$-equivalence classes of P, and with the following universal property: for any poset R and any morphism $g : P \to R$ such that g is constant on $\sim$-equivalence classes of P, there is a unique morphism $h : T \to R$ such that $g = h \circ f$. Such a pair (T, f) exists and is unique up to a unique isomorphism of posets. It is constructed as follows: consider first the set T' of $\sim$-equivalence classes of P, with the obvious mapping $f' : P \to T'$, and equip T' with the smallest preorder relation making f' a morphism of preorders. A quotient T is then given as the collapse poset of T', with $f : P \to T$ the obvious mapping obtained by composition $P \to T' \to T$.

Since quotients are unique up to a unique isomorphism, we freely talk of *the* quotient of a poset P with respect to an equivalence relation $\sim$. We denote the quotient by $P/\sim$, and we insist that this notation does not necessarily denote the mere preorder of $\sim$-equivalence classes of P.

3.3 Completion and quotients: a counterexample. Since we both have the notions of quotient and of ω-completion available for posets, it is worth knowing if, in short, both operations commute. Formulated like this, the question is not precise enough, since if we start from an equivalence relation on a poset P, we first need to extend the equivalence relation to the ω-completion Q of P in order to compare quotient of completion and completion of quotient.

Before giving a general way to extend an equivalence relation to a completion, we can observe that a canonical way is obviously given in the case where P coincides with its completion, *i.e.*, according to Theorem 2.3, when P is ω-chain complete. This already gives a general negative answer to the above question. Consider indeed the poset P depicted at left on Figure 1. An equivalence relation $\sim$ on P is depicted by curved lines labeled by a "$\sim$" symbol. We have added a bottom element to show that bottom elements do not help for the issue we are concerned with. On the one hand, this poset is ω-chain complete, and so P

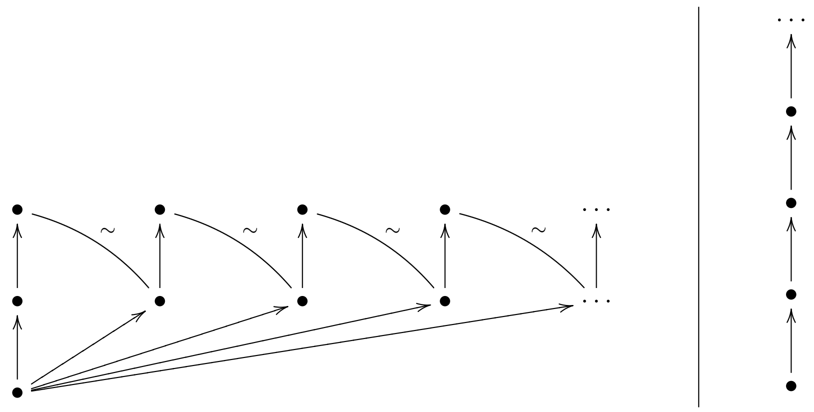

Figure 1: The quotient of the completion is *not* the completion of the quotient.

coincides with its completion Q. Observe also that P is a strict poset. On the other hand, the quotient $G = P/\sim$ of P is depicted at right on Figure 1, and it is not $\boldsymbol{\omega}$-chain complete. Hence there is no chance that the completion of G coincides with a quotient of $Q = P$.

We now embark on extending equivalence relations on posets to their completions. From now on, we will focus on *strict posets* (see Definition 2.7).

3.4 Definition. — *Let $(P, \sqsubseteq)$ be a poset equipped with an equivalence relation $\sim$, and let $(G, \preccurlyeq)$ be the quotient. Let $P \to Q$ be the $\boldsymbol{\omega}$-completion of P. For $x, y \in Q$, we define $x \lesssim y$ if and only if there are two $\boldsymbol{\omega}$-chains D_x and D_y of P such that $\sup_Q D_x = x$, $\sup_Q D_y = y$, and:*

$$\forall u \in D_x, \quad \exists v \in D_y, \quad p(u) \preccurlyeq p(v)\,, \tag{1}$$

where $p : P \to G$ stands for the canonical projection.

One can see in the above definition a variant of so-called Egli-Milner order [4], which arose in domain-theoretic models of nondeterminism.

3.5 Lemma. — *We keep the notations of Definition 3.4, and we assume that the $\boldsymbol{\omega}$-completion $P \to Q$ is strict. Then for any $x, y \in Q$, and for any $\boldsymbol{\omega}$-chains D_x and D_y of $(P, \sqsubseteq)$ with $\sup_Q D_x = x$ and $\sup_Q D_y = y$, if $x \lesssim y$, then the property stated in Eq.* (1) *holds.*

Proof. — Let $x, y \in P$ with $x \lesssim y$, and let E_x and E_y as in Eq. (1). Let D_x and D_y be as in the statement of the lemma, and pick $u \in D_x$. By $\boldsymbol{\omega}$-compactness of u, since $u \sqsubseteq \sup_Q E_x$, there is an element $s \in E_x$ with $u \sqsubseteq s$. There is then a t in E_y with $p(s) \preccurlyeq p(t)$. By $\boldsymbol{\omega}$-compactness of t, there is $v \in D_y$ with $t \sqsubseteq v$. We have then $p(u) \preccurlyeq p(s) \preccurlyeq p(y) \preccurlyeq p(v)$. ∎

3.6 Corollary. — *We keep the notations of Definition 3.4, and we assume that the $\boldsymbol{\omega}$-completion $P \to Q$ is strict. Let $x \in P$ and $y \in Q$ such that $x \lesssim y$.*

Then for any $\boldsymbol{\omega}$-chain D_y of P such that $\sup_Q D_y = y$*, there exists $z \in D_y$ such that $p(x) \preccurlyeq p(z)$.*

Proof. — We apply Lemma 3.5 to $\boldsymbol{\omega}$-chains $\{x\}$ and D_y. ∎

3.7 Lemma. — *We keep the notations of Definition 3.4, and we assume that the $\boldsymbol{\omega}$-completion $P \to Q$ is strict. Then relation $\lesssim$ is a preorder on Q. If $x, y \in P$, then $x \lesssim y$ if and only if $p(x) \preccurlyeq p(y)$ (in G).*

Proof. — That $\lesssim$ is a preorder on Q follows from directly Lemma 3.5.

Now let $x, y \in P$. If $x \lesssim y$, then by Corollary 3.6 applied to x and $D_y = \{y\}$, it follows that $p(x) \preccurlyeq p(y)$. Conversely, if $p(x) \preccurlyeq p(y)$, then considering $D_x = \{x\}$ and $D_y = \{y\}$ immediately shows by the very definition that $x \lesssim y$. ∎

3.8 Definition. — *Let $(P, \sqsubseteq) \to (Q, \sqsubseteq)$ be a strict $\boldsymbol{\omega}$-completion, and assume that P is equipped with an equivalence relation $\sim$. Let $(G, \preccurlyeq)$ denote the quotient of P with respect to $\sim$. Using the preorder relation $\lesssim$ on Q introduced in Definition 3.4, we define the* pseudo-completion *of the quotient G as the poset $(H, \precsim)$, collapse of $(Q, \lesssim)$.*

By this way, we have extended the equivalence relation $\sim$ from P to its $\boldsymbol{\omega}$-completion Q. Indeed, the equivalence relation $\approx$ canonically associated to preorder $\lesssim$ and defined by $\approx = \lesssim \cap (\lesssim)^{-1}$ is an extension to Q of the equivalence relation $\sim$ on P.

In general, the pseudo-completion is *not* the $\boldsymbol{\omega}$-completion of G, as our discussion in § 3.3 shows.

3.9 Lemma. — *With the notations of Definition 3.8, there is a commutative square in category* PO*:*

$$\begin{tikzcd}
(P, \sqsubseteq) \arrow[r] \arrow[d, "p"] & (Q, \sqsubseteq) \arrow[d, "q"] \\
(G, \preccurlyeq) \arrow[r] & (H, \precsim)
\end{tikzcd}$$

where both horizontal arrows are embeddings, and $q : H \to Q$ is the canonical projection.

Proof. — The existence of a non decreasing, injective arrow $(G, \preccurlyeq) \to (H, \precsim)$ follows from Lemma 3.7. ∎

Since $(G, \preccurlyeq)$ is a subobject of $(H, \precsim)$, we simply denote by $(H, \preccurlyeq)$ the pseudo-completion of G from now on.

4 Preordered and Partially Ordered Semigroups

4.1 Semigroups. In this paper we call *semigroup* a set S equipped with an associative internal composition $(x, y) \in S \times S \mapsto x \cdot y \in S$, for which there exists a neutral element, denoted by ϵ and which is necessarily unique. We denote by SG the category of semigroups with semigroup morphisms as morphisms.

4.2 Congruences. We say that an equivalence relation $\mathscr{R}$ on a semigroup S is a *congruence* if it is compatible with the semigroup composition. That is to say, for all $x, y, z \in S$, we have:

$$x \mathscr{R} y \Rightarrow \begin{cases} (x \cdot z) \mathscr{R} (y \cdot z), & \text{and} \\ (z \cdot x) \mathscr{R} (z \cdot y). \end{cases}$$

The *quotient* semigroup is then the set of equivalence classes modulo $\mathscr{R}$, equipped with the induced semigroup structure.

4.3 Preordered and ordered semigroups. We say that a semigroup S is *preordered* if S is equipped with a preorder relation $\sqsubseteq$ such that:

1. $\forall x \in S, \quad \epsilon \sqsubseteq x$;
2. $\forall x, y, u \in S, \quad x \sqsubseteq y \Rightarrow u \cdot x \sqsubseteq u \cdot y$.

We say the semigroup S is *ordered* if it is preordered by a relation which is a partial order.

The prototypical example of ordered semigroup is the free semigroup $\Sigma^\star$, which consists of finite words on alphabet Σ, equipped with the prefix order.

4.4 An adjunction for preordered semigroups. Let PreOSG denote the category with preordered semigroups as objects, and with mappings between them that are both morphisms of semigroups and of preorders as arrows. Then the forgetful functor $\mathsf{PreOSG} \to \mathsf{SG}$ is the right member of an adjunction pair, the left member of which sends a semigroup S to the preordered semigroup with same underlying semigroup S, and with preorder relation $\sqsubseteq$ defined by:

$$\forall x, y \in S, \quad x \sqsubseteq y \iff \exists z \in S, \quad y = x \cdot z. \tag{2}$$

We leave to the reader to check this defines indeed an adjunction, which is furthermore a reflection.

4.5 Adjunction for partially ordered semigroups. Let POSG denote the category of partially ordered semigroups, defined as the full subcategory of PreOSG the objects of which are partially ordered semigroups. There is then an obvious adjunction $\mathsf{POSG} \leftrightarrows \mathsf{PreOSG}$, obtained by restriction of the adjunction $\mathsf{PO} \leftrightarrows \mathsf{PreO}$. Since adjunctions compose, we obtain thus an adjunction $\mathsf{POSG} \leftrightarrows \mathsf{SG}$.

This adjunction justifies the following definition.

4.6 Definition. — *We say that a semigroup S is* naturally ordered *if S is obtained as $S = \mathsf{R}(T)$, where $\mathsf{R} : \mathsf{SG} \to \mathsf{POSG}$ is the left member of the above adjunction, and T is some semigroup. It is thus a semigroup preordered as in Eq.* (2)*, with the additional property that the preordering is actually a partial order.*

4.7 Congruences and partially ordered semigroups. Let S be a semigroup equipped with a congruence $\mathscr{R}$. Let $\pi : S \to S/\mathscr{R}$ denote the quotient semigroup. Considering the action of the functor $\mathsf{R} : \mathsf{SG} \to \mathsf{POSG}$ yields the following commutative diagram, in the category of semigroups:

$$\begin{tikzcd} S \arrow[r] \arrow[d, "\pi"'] & \mathsf{R}(S) \arrow[d, "\mathsf{R}(\pi)"] \\ S/\mathscr{R} \arrow[r] & \mathsf{R}(S/\mathscr{R}) \end{tikzcd} \tag{3}$$

Denote $S' = \mathsf{R}(S)$, $\pi' = \mathsf{R}(\pi)$, and consider the equivalence relation $\mathscr{C}$ on S' defined by $x \mathscr{C} y \iff \pi'(x) = \pi'(y)$ for $x, y \in S'$. Then $\mathscr{C}$ is a congruence on semigroup S', and there is a natural isomorphism of semigroups $S'/\mathscr{C} \to \mathsf{R}(S/\mathscr{R})$. By construction, $S'/\mathscr{C}$ when partially ordered with the natural partial order, is also the quotient in the sense of partial orders of S' with respect to $\mathscr{C}$. We are thus brought to introduce the following definition.

4.8 Definition. — *Let S be a semigroup equipped with a congruence $\mathscr{R}$. Keeping the notations of § 4.7, we say that the congruence $\mathscr{C}$ defined on the naturally partially ordered semigroup $\mathsf{R}(S)$ is the* natural ordering congruence *associated with $\mathscr{R}$.*

If S is a naturally partially ordered semigroup, so that we may identify S and $\mathsf{R}(S)$, we say that a congruence $\mathscr{R}$ on S is compatible with the natural ordering *on S if the associated natural ordering congruence $\mathscr{C}$ coincides with $\mathscr{R}$.*

5 Completion of Quotient Semigroups

The aim of this section is to continue the study of Section 3, by particularizing to naturally ordered semigroups. We will show that in this case the pseudo-completion that we have defined in § 3.8 coincides with the $\boldsymbol{\omega}$-completion.

5.1 General framework and notations. Throughout this section, we consider a semigroup S naturally ordered, equipped with a congruence relation $\mathscr{R}$ that we assume compatible with the natural ordering (see Definitions 4.6 and 4.8 respectively). The ordering relation is denoted $\sqsubseteq$. We denote by G

the quotient semigroup $S/\mathscr{R}$, which is naturally ordered by relation $\preccurlyeq$. We assume that the poset $(S, \sqsubseteq)$ has a strict $\boldsymbol{\omega}$-completion. We denote by $(Q, \sqsubseteq)$ and by $(H, \preccurlyeq)$ respectively the $\boldsymbol{\omega}$-completion of $(S, \sqsubseteq)$ and the pseudo-completion of $(G, \preccurlyeq)$. Denoting by $p : S \to G$ and by $q : Q \to H$ the canonical projections, we have the following commutative diagram, where horizontal arrows are embeddings of posets:

$$\begin{tikzcd} (S, \sqsubseteq) \arrow[r] \arrow[d, "p"'] & (Q, \sqsubseteq) \arrow[d, "q"] \\ (G, \preccurlyeq) \arrow[r] & (H, \preccurlyeq) \end{tikzcd}$$

Finally, we denote by $\precsim$ the preorder on Q introduced in Definition 3.4, and we recall that $(H, \preccurlyeq)$ is defined as the collapse poset associated with $(Q, \precsim)$.

The goal of this section is to show the following result. It provides us with a description of the completion of a quotient $G = S/\mathscr{R}$, in terms of the completion of S. Typically, if $S = \Sigma^\star$ is a free semigroup, this description is fully operational since the completion $\overline{\Sigma}$ of $\Sigma^\star$ is just the poset of countable infinite words on Σ.

5.2 Theorem. — *Let S be a semigroup naturally ordered, equipped with a congruence relation $\mathscr{R}$ compatible with the natural ordering. Assume that S is a strict poset. Then the quotient $G = S/\mathscr{R}$ is a strict poset, the $\boldsymbol{\omega}$-completion of which coincides with its pseudo-completion.*

We begin with some lemmas.

5.3 Lemma. — *Let x be an element of S, and let y be an element of Q. Assume that $q(x) \preccurlyeq q(y)$. Then for every $\boldsymbol{\omega}$-chain D of S such that $\sup_Q D = y$, there exists $z \in D$ such that $p(x) \preccurlyeq p(z)$.*

Proof. — This is an immediate consequence of Corollary 3.6. ∎

5.4 Lemma. — *Let $(x_n)_{n \geq 0}$ be a sequence of elements in G such that:*

$$\forall m, n \geq 0, \quad n \leq m \Rightarrow x_n \preccurlyeq x_m\,.$$

Then there exists a sequence $(y_n)_{n \geq 0}$ of elements in S such that $p(y_n) = x_n$ for all $n \geq 0$, and such that:

$$\forall m, n \geq 0, \quad n \leq m \Rightarrow y_n \sqsubseteq y_m\,.$$

Proof. — Let $(x_n)_{n \geq 0}$ be a sequence as in the statement of the lemma. We construct the sequence $(y_n)_{n \geq 0}$ by induction on $n \geq 0$. Pick any element $y_0 \in S$ such that $p(y_0) = x_0$. Assume that $y_0, \ldots, y_n$ have been constructed, for $n \geq 0$, such that $p(y_i) = x_i$ for $i = 0, \ldots, n$, and such that $y_i \sqsubseteq y_{i+1}$ for $i = 0, \ldots, n-1$. Pick an element $z \in S$ such that $p(z) = x_{n+1}$. Then there exists an element

$h \in S$ such that $z \mathscr{R} (y_n \cdot h)$. Hence choosing $y_{n+1} = y_n \cdot h$ we have $p(y_{n+1}) = p(z) = x_{n+1}$, and $y_n \sqsubseteq y_{n+1}$. The induction is complete. ❙

5.5 Lemma. — *Any $\boldsymbol{\omega}$-chain in G has a lub in H.*

Proof. — Let C be a $\boldsymbol{\omega}$-chain in G. Without loss of generality, we assume that there is a sequence $(x_n)_{n\geq 0}$ in S such that $C = \{x_n,\, n \geq 0\}$, and such that $i \leq j \Rightarrow x_i \preccurlyeq x_j$ for all $i, j \geq 0$. Indeed, if any chain of this kind has a *lub* in H, then any chain of G has a *lub* in H. Let $(y_n)_{n\geq 0}$ be a sequence in S associated with $(x_n)_{n\geq 0}$ as in Lemma 5.4. Then $\{y_n,\, n \geq 0\}$ has a *lub* in Q, say $z = \sup_Q\{y_n,\, n \geq 0\}$. We put $x = q(z)$, and we show that x is the *lub* of $\{x_n,\, n \geq 0\}$ in H.

1. It is obvious that x is an upper bound of $C = \{x_n,\, n \geq 0\}$.
2. Let $r \in H$ be any upper bound of C, let $s \in Q$ such that $r = q(s)$ and let D be a $\boldsymbol{\omega}$-chain in S such that $s = \sup_Q D$. Then we have $x_n = p(y_n) \preccurlyeq q(s)$ for any $n \geq 0$. Thus, by Lemma 5.3, it follows that there is an element $c \in D$ such that $p(y_n) \preccurlyeq p(c)$. Since $s = \sup_Q D$ and $z = \sup_Q\{y_n,\, n \geq 0\}$, it follows by the very definition of relation $\lesssim$ that $z \lesssim s$ holds, and thus $x = q(z) \preccurlyeq q(s) = r$.

This shows that $x = \sup_H C$. The proof is complete. ❙

5.6 Lemma. — *Poset $(H, \preccurlyeq)$ is $\boldsymbol{\omega}$-chain complete.*

Proof. — We combine a diagonal argument with the result of Lemma 5.5 to prove that any chain in H has a *lub*. For this, let C be a $\boldsymbol{\omega}$-chain in H. We assume without loss of generality that there is a sequence $(x_n)_{n\geq 0}$ in H such that $X = \{x_n,\, n \geq 0\}$, and such that $i \leq j \Rightarrow x_i \preccurlyeq x_j$ for all integers $i, j \geq 0$. For, if any chain of this kind has a *lub* in H, then any $\boldsymbol{\omega}$-chain in H has a *lub*. We construct by induction on integer $i \geq 0$ an array $(x_i^j)_{i,j\geq 0}$ of elements in G with the following properties:

1. For all $i \geq 0$, sequence $(x_i^j)_{j\geq 0}$ is increasing, and $\sup_H\{x_i^j,\, j \geq 0\} = x_i$.
2. We have $x_i^j \preccurlyeq x_{i+1}^j$ for all integers $i, j \geq 0$.

The induction is initialized by choosing any increasing sequence $(x_0^j)_{j\geq 0}$ such that

$$\sup_H\{x_0^j,\, j \geq 0\} = x_0 \,.$$

Assume that sequences $(x_i^j)_{j\geq 0}$ have been constructed for $i = 0, \ldots, n$, with Property 1 satisfied for $i = 0, \ldots, n$ and Property 2 satisfied for $i = 0, \ldots, n-1$. Let $(z_m)_{m\geq 0}$ be an increasing sequence in G such that

$$\sup_H\{z_m,\, m \geq 0\} = x_{n+1} \,.$$

Let j be any integer. Since $x_n \preccurlyeq x_{n+1}$, in particular $x_n^j \preccurlyeq x_{n+1}$, and therefore it follows from Lemma 5.3 that there exists an integer m such that $x_n^j \preccurlyeq z_m$. We put $x_{n+1}^j = z_{m \vee j}$. Sequence $(x_{n+1}^j)_{j \geq 0}$ thus defined is increasing, and satisfies $x_n^j \preccurlyeq x_{n+1}^j$ for all $j \geq 0$. Hence Property 2 holds for $i = n$. Furthermore $z_k \preccurlyeq x_{n+1}^k \preccurlyeq x_{n+1}$ for all $k \geq 0$, and therefore

$$\sup_H \{x_{n+1}^j,\, j \geq 0\} = x_{n+1}\,.$$

Hence Property 1 is satisfied for $i = n + 1$, and the induction is complete.

We consider now the sequence $(c_n)_{n \geq 0}$ in G defined by $c_n = x_n^n$. Properties 1 and 2 above imply that $(c_n)_{n \geq 0}$ is increasing, since

$$c_n = x_n^n \preccurlyeq x_{n+1}^n \preccurlyeq x_{n+1}^{n+1} = c_{n+1}$$

for every integer $n \geq 0$. It follows from Lemma 5.5 that $(c_n)_{n \geq 0}$ has a *lub* in H, say $c \in H$. We show that

$$c = \sup_H \{x_n\,, n \geq 0\}\,. \tag{4}$$

First, we show that c is an upper bound of $\{x_n,\, n \geq 0\}$. Indeed, for any integer $m \geq 0$, and for any integer $j \geq m$, we have $x_m^j \preccurlyeq x_j^j$ and therefore

$$x_m = \sup_H \{x_m^j,\, j \geq 0\} \preccurlyeq \sup_H \{x_j^j,\, j \geq 0\} = c\,.$$

Then we show that c is the least upper bound of $\{x_n,\, n \geq 0\}$. For this, let r be any upper bound of $\{x_n,\, n \geq 0\}$. Then for any integer $n \geq 0$, we have $c_n = x_n^n \preccurlyeq x_n \preccurlyeq r$, and thus $c \preccurlyeq r$. This shows Eq. (4), and completes the proof of the lemma. ∎

5.7 Proof of Theorem 5.2. We have shown in Lemma 5.6 that H is $\boldsymbol{\omega}$-chain complete. The same diagonal argument used in the proof of this lemma shows that H satisfies the universal property of $\boldsymbol{\omega}$-chain completions. It remains only to show that the completion $G \to H$ is strict. The fact that any element in G is $\boldsymbol{\omega}$-chain compact follows from Lemma 5.3. And, by construction, any element of H is obtained as the *lub* of a $\boldsymbol{\omega}$-chain in G. The proof is complete. ∎

6 Action of a Semigroup on its Completion. Completions as Semigroups

6.1 Semigroup action and semigroup structure for completions. The aim of this section is to explore the ability of endowing the $\boldsymbol{\omega}$-chain completion of a semigroup with a semigroup structure. A first step is to consider a semigroup action of a semigroup on its $\boldsymbol{\omega}$-chain completion. A second step is then to extend this action to a semigroup structure, since the semigroup is embedded into its $\boldsymbol{\omega}$-chain completion—the definition of all these notions will be recalled below.

Furthermore, since we have equipped $\boldsymbol{\omega}$-chain completions with its $\boldsymbol{\omega}$-Scott topology (see § 2.8), it is also natural to inspect the topological counterpart of these notions—topological semigroup actions and topological semigroup structures. However, when considering the $\boldsymbol{\omega}$-Scott topology, there is, in general, no topological semigroup structure on the $\boldsymbol{\omega}$-chain completion that would extend the original semigroup structure on the semigroup. We will be brought in next section to introduce *strongly ordered semigroups* in order to obtain $\boldsymbol{\omega}$-Scott topological semigroups.

We first recall some standard definitions.

6.2 Semigroup action and topological semigroup action. Let S be a semigroup, and let X be a set. We say that S *acts on* X if there is a mapping $f_s : X \to X$ associated with each element $s \in S$, such that $f_\epsilon = \mathrm{Id}_X$, and $f_{(s\cdot t)} = f_s \circ f_t$ for every $s, t \in S$. We then simply write $s \cdot x$ instead of $f_s(x)$, for $s \in S$ and $x \in X$, so that we have $(s \cdot t) \cdot x = s \cdot (t \cdot x)$ for $s, t \in S$ and $x \in X$, and $\epsilon \cdot x = x$ for every $x \in X$.

The internal composition defines for every semigroup S a natural action on itself, making the "$\cdot$" notation consistent.

Assume that X is a topological space and that an action of a semigroup S on X is defined. We say that the action is *topological* if all mappings $x \in X \mapsto s \cdot x$ are continuous, for $s \in S$.

6.3 Topological action of a semigroup on its $\boldsymbol{\omega}$-completion. We consider a partially ordered semigroup $(G, \preccurlyeq)$, with $\boldsymbol{\omega}$-chain completion $(H, \preccurlyeq)$, and we assume that $G \to H$ is a strict $\boldsymbol{\omega}$-chain completion—in particular, we may consider the $\boldsymbol{\omega}$-completion $G \to H$ given by Theorem 5.2.

Let $s \in G$, $x \in H$, and let D be a $\boldsymbol{\omega}$-chain in G such that $x = \sup_H D$. Denoting by $s \cdot D$ the following subset:

$$s \cdot D = \{s.y,\ y \in D\},$$

it is obvious from the definition in § 4.3 to check that $s \cdot D$ is a $\boldsymbol{\omega}$-chain in G. We put

$$s \cdot x = \sup_H (s \cdot D). \tag{5}$$

This definition is justified by the following result.

6.4 Proposition. — *We assume that the $\boldsymbol{\omega}$-completion $G \to H$ is strict. Then the element $s \cdot x \in H$ defined by Equation* (5) *does not depend on the $\boldsymbol{\omega}$-chain D. This defines a semigroup action of G on H that extends the semigroup structure of G via the embedding $G \to H$. Furthermore, the semigroup action is topological when H is equipped with the $\boldsymbol{\omega}$-Scott topology.*

Proof. — The techniques we use here are the same involved in § 5, we thus only sketch the proof of the proposition.

The fact that $\sup_H(s \cdot D) = \sup_H(s \cdot E)$ for any $s \in G$ and for any $\boldsymbol{\omega}$-chains D and E in G such that $\sup_H D = \sup_H E$ rests on the assumption that the extension $G \to H$ is strict. It is then routine verification to check that $s \cdot x$ thus defined gives raise to a semigroup action that extends the semigroup structure of G.

To see that the semigroup action is $\boldsymbol{\omega}$-Scott continuous, we have to show that, for any $s \in S$, mapping $x \in H \mapsto s \cdot x$ is increasing, and that for any $\boldsymbol{\omega}$-chain $(x_n)_{n\geq 0}$ in H, we have:

$$\sup_H\{s \cdot x_n,\, n \geq 0\} = s \cdot \bigl(\sup\{x_n\,,\ n \geq 0\}\bigr)\,. \tag{6}$$

For this, we proceed in two steps. We first show that the restricted mapping $x \in G \mapsto s \cdot x$ is a morphism in category PO'. It is obviously increasing, and we check that Equation (6) is valid if $(x_n)_{n\geq 0}$ is a sequence in G. The universal property of $\boldsymbol{\omega}$-chain completion H implies that this mapping extends uniquely to a $\boldsymbol{\omega}$-chain continuous mapping $H \to H$. The latter can only be our mapping $x \in H \mapsto s \cdot x$, by the very definition (5). This completes the proof. ▮

6.5 A semigroup structure on $\boldsymbol{\omega}$-chain completions. Having defined this semigroup action of G on H, we may ask whether it can be extended to a semigroup structure on H. Indeed, we can just put, for any $x \in H$:

$$s \cdot x = \begin{cases} \text{defined as above in (5),} & \text{if } s \in G \\ s, & \text{if } s \in H \setminus G \end{cases} \tag{7}$$

Thanks to the result of Proposition 6.4, this defines a semigroup structure on H, that extends the semigroup structure on G. We discuss below the relevance of this structure with respect to the notion of topological semigroups.

6.6 Topological semigroups. Let S be a semigroup equipped with a topology. We say that S is a *topological semigroup* if the mapping $(s,t) \in S\times S \mapsto s\cdot t$ is continuous, with $S \times S$ equipped with the product topology.

For partially ordered semigroups, it is a fact that the mappings $s \in S \mapsto s\cdot t$, for $t \in S$, is not an increasing mapping in general. This is typically the case for naturally ordered free semigroups, as the reader may easily check. Hence, when thinking of $\boldsymbol{\omega}$-chain completion of semigroups, it is hopeless to try equipping the $\boldsymbol{\omega}$-chain completion of a semigroup with a $\boldsymbol{\omega}$-Scott continuous semigroup structure, that would extend the original semigroup structure. In particular, the definition (7) is frustrating from this point of view, since the element $x \cdot y$ for $x \in H \setminus G$ and $y \in H$ is *not* defined by continuous approximations, but rather by an *ad hoc* formula, only retained for its associativity features. This contrasts with the definition (5) of $s\cdot x$ for $s \in G$ and $x \in H$. Next section gives some insight on a more favorable situation in this respect.

7 Strongly Ordered Semigroups

We begin with the definition.

7.1 Definition. — *Let S be a partially ordered semigroup. We say that S is* strongly ordered *if, for every element $x \in S$, the right multiplication by x, defined by*

$$r_x : y \in S \mapsto y \cdot x \,,$$

is increasing.

As already observed in § 6.6, ordered semigroups are not strongly ordered in general, as the example of free semigroups shows. Examples of strongly ordered semigroups are given in §§ 7.3, 7.4.

7.2 Properties of strongly ordered semigroups. Let $(S, \preccurlyeq)$ be a strongly ordered semigroup. Then any two elements $x, y \in S$ have $x \cdot y$ and $y \cdot x$ as common upper bounds. Indeed, $x \preccurlyeq x \cdot y$ always holds for ordered semigroups. Here moreover, since the ordering is strong, from $\epsilon \preccurlyeq x$ we also get $y = r_y(\epsilon) \preccurlyeq r_y(x) = x \cdot y$, showing that $x \cdot y$ is an upper bound of x and y. By symmetry, it follows that $y \cdot x$ is also an upper bound of x and y.

Obviously, commutative ordered semigroups are strongly ordered. To construct less trivial examples however, we make the following observation. Assume that S is naturally ordered. Then saying the ordering is strong is like assuming the existence of pseudo-conjugates. Indeed, for any $x, y \in S$, we have $y \preccurlyeq x \cdot y$, as we have just noted. Hence there exists an element $r \in S$ such that $x \cdot y = y \cdot r$. If S were embedded in a group Γ, then r would be unique, given by $r = y^{-1} \cdot x \cdot y$. This observation gives us a hint to construct examples of strongly ordered semigroups.

7.3 Examples. Examples of strongly ordered semigroups are found in the algebraic theory of language, where syntactic semigroups of regular languages are finite and strongly ordered semigroups [8]. Since they are finite however, syntactic semigroups are not relevant here, so we present examples given by generators and relations.

Let $(S, \sqsubseteq)$ be the naturally preordered semigroup with three generators a, b and c, subject to the following relations:

$$a \cdot b \sim b \cdot c \sim c \cdot a \,, \qquad b \cdot a \sim c \cdot b \sim a \cdot c \,. \tag{8}$$

Observe that the length of all words belonging to an equivalence class is the same, whence the definition of the length of an element in S. Length is then an additive function with respect to the semigroup composition, and ϵ is the only element in S of length zero. This implies in particular that the natural preordering $\sqsubseteq$ is a partial ordering, and that no element of the quotient semigroup has an inverse.

We claim that S thus defined is strongly ordered. Since S is generated by $\{a, b, c\}$, it is enough for this to show that the right multiplications r_x are increasing for $x = a, b, c$. By symmetry, we assume without loss of generality that $x = a$. We are thus brought to show that

$$y \cdot a \sqsubseteq y \cdot z \cdot a \tag{9}$$

for any $y, z \in S$. It is enough to show Equation (9) for $y, z \in \{a, b, c\}$, and thus to find the "pseudo-conjugates" r such that $y \cdot z \cdot a = y \cdot a \cdot r$ for such elements y, z. Define the following elements:

$$r_a = a, \quad r_b = c, \quad r_c = b\,.$$

It is then routine verification to check, using relations (8) and (9), that we have for any $y \in \{a, b, c\}$

$$y \cdot z \cdot a = y \cdot a \cdot r_z\,.$$

This shows that Equation (9) holds, and by this way we have shown that S is strongly ordered.

7.4 Generalizing the example. More generally, let $\Sigma^\star$ be a free semigroup, we consider a mapping $\Phi : \Sigma^\star \to \operatorname{End}\Sigma^\star$, where $\operatorname{End}\Sigma^\star$ denotes the semigroup of endomorphisms $\Sigma^\star \to \Sigma^\star$, such that Φ is an anti-morphism of semigroups, *i.e.*, $\Phi(\epsilon) = \mathrm{Id}_{\Sigma^\star}$, and $\Phi(u \cdot v) = \Phi(v) \circ \Phi(u)$. Then we consider the quotient of the free semigroup $\Sigma^\star$ under the relations

$$x \cdot y \sim y \cdot \Phi(y)(x),$$

for $x, y \in \Sigma$. This yields a semigroup that we preorder by the natural preorder, the collapse of which is strongly ordered. The above example in § 7.3 were constructed by this way, with Φ defined as follows:

$$\begin{array}{lll}
\Phi(a)(a) = a, & \Phi(a)(b) = c, & \Phi(a)(c) = b, \\
\Phi(b)(a) = c, & \Phi(b)(b) = b, & \Phi(b)(c) = a, \\
\Phi(c)(a) = b, & \Phi(c)(b) = a, & \Phi(c)(c) = c.
\end{array}$$

We give below in Theorem 7.6 a result on the $\boldsymbol{\omega}$-chain completion of strongly ordered semigroups. It is convenient to introduce countable directed sets, that we call $\boldsymbol{\omega}$-directed sets, instead of $\boldsymbol{\omega}$-chains.

7.5 Lemma. — *Let D, D' be two directed sets of a strongly ordered semigroup S. Then the set $D \cdot D'$ defined by:*

$$D \cdot D' = \{d \cdot d',\, (d, d') \in D \times D'\} \tag{10}$$

is a directed subset of S.

Proof. — Since semigroup S is strongly preordered, we check that for any two pairs (d, d') and (e, e') in $D \times D'$, the elements $d \cdot d'$ and $e \cdot e'$ have $x \cdot x'$ as an upper bound, where x and x' are upper bounds in D and in D' respectively of d and d' and of e and e'. ∎

7.6 Theorem. — *Let $(G, \preccurlyeq)$ be a strongly ordered semigroup, that we assume to have a strict extension $G \to H$. Equip H with the $\boldsymbol{\omega}$-Scott topology.*

Then there is a unique continuous semigroup structure on H that extends the semigroup structure of G. This semigroup structure on H satisfies for any two $\boldsymbol{\omega}$-directed subsets D and D' in H:

$$\sup_H(D \cdot D') = (\sup_H D) \cdot (\sup_H D') \,. \tag{11}$$

Proof. — Let $x, y \in H$. Pick D_x and D_y two $\boldsymbol{\omega}$-directed subsets of G such that

$$\sup_H D_x = x, \quad \sup_H D_y = y \,.$$

If there exists a continuous semigroup structure on H that extends the semigroup structure on G, the element $x \cdot y$ must be defined by:

$$x \cdot y = \sup_H(D_x \cdot D_y) \,, \tag{12}$$

where $D_x \cdot D_y$ is defined in (10), and is directed thanks to Lemma 7.5. It is then routine to check that the internal composition given by (12) is well defined, *i.e.*, only depends on x and y, and is an associative composition that extends the internal composition in G, with ϵ as identity element.

What remains to show is that $\Phi : (x, y) \in H \times H \mapsto x \cdot y$ is continuous. As we have seen in § 2.9, and since the completion $G \to H$ is assumed to be strict, the $\boldsymbol{\omega}$-Scott topology is generated by subsets $\uparrow u$, for u ranging over G. It is thus enough to show that $U = \Phi^{-1}(\uparrow u)$, for $u \in G$, is open in the product topology on $H \times H$ to get that Φ is continuous. Let $(x, y) \in U$, and consider D_x and D_y two $\boldsymbol{\omega}$-chain in G such that $x = \sup_H D_x$ and $y = \sup_H D_y$. Then we have

$$u \preccurlyeq x \cdot y = x \cdot \sup_H D_y = \sup_H(x \cdot D_y) \,,$$

and since u is $\boldsymbol{\omega}$-compact in H, this implies that there exists $w \in D_y$ such that $u \preccurlyeq x \cdot w$. Since $x \cdot w = \sup_H(D_x \cdot z)$, again by $\boldsymbol{\omega}$-compactness of u, there is an element $v \in D_x$ such that $u \preccurlyeq v \cdot w$. Then $V = \uparrow v \times \uparrow w$ is open in the product topology on $H \times H$, and satisfies:

$$(x, y) \in V, \quad V \subseteq \Phi^{-1}(\uparrow u) \,.$$

Since such an open set V exists for any $(x, y) \in \Phi^{-1}(\uparrow u)$, this shows that $\Phi^{-1}(\uparrow u)$ is open, and thus that Φ is continuous, as requested.

Equation (11) now follows for any $\boldsymbol{\omega}$-directed subsets D and D' of H from the continuity of the semigroup structure on H. The proof is complete. ∎

8 Conclusion

We have studied the $\boldsymbol{\omega}$-completion of naturally ordered semigroups. In particular we have studied the case of quotient of a semigroup S with a strict $\boldsymbol{\omega}$-completion. In this case, quotient and completion commute, which provides us with a concrete, operational description of the $\boldsymbol{\omega}$-completion of the quotient.

This applies in particular for quotients of free semigroups. We have by this way demonstrated the relevance of the notion of strict $\boldsymbol{\omega}$-completion.

As applications of the universal property of completions, we have studied the topological semigroup structure that can be given to the completion. Equipping the $\boldsymbol{\omega}$-completion with its $\boldsymbol{\omega}$-Scott topology, there is always a topological semigroup action of the semigroup on its $\boldsymbol{\omega}$-completion, and it can be extended to a semigroup structure on the completion. However, this semigroup structure cannot be topological with respect to the $\boldsymbol{\omega}$-Scott topology in general. This has motivated us to study semigroups where the partial order relation is both a left and a right congruence, called strongly ordered semigroups in this paper. The $\boldsymbol{\omega}$-completion of a strongly ordered semigroup can be endowed with a topological semigroup structure that extends the semigroup structure on the originating semigroup.

Future work we plan would use the results given here in a different field. We wish to study increasing random walks on partially ordered semigroups. We believe that it would be interesting to compare the $\boldsymbol{\omega}$-completion studied in this paper, with other boundary constructions that come from measure theory, namely stationary and Poisson boundaries. We hope to find interesting results concerning the comparison of both kinds of boundaries.

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